ON GENERALISED KNESER COLOURINGS

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Abstract. There are two possible definitions of the "s-disjoint r-uniform Kneser hypergraph" of a set system $\mathcal{T}$: The hyperedges are either r-sets or r-multisets. We point out that Ziegler's (combinatorial) lower bound on the chromatic number of an s-disjoint r-uniform Kneser hypergraph only holds if we consider r-multisets as hyperedges. We give a new proof of his result and show by example that a similar result does not hold if one considers r-sets as hyperedges.

In case of r-sets as hyperedges and $s \geq 2$ the only known lower bounds are obtained from topological invariants of associated simplicial complexes if r is a prime or the power of prime. This is also true for arbitrary r-uniform hypergraphs with r-sets or r-multisets as hyperedges as long as r is a power of a prime.

1. Introduction

During the last 25 years, topological methods have been successfully applied to obtain lower bounds of the chromatic number of graphs and hypergraphs. It all took off with Lovász' seminal paper [7] were he proved that the connectivity of the neighbourhood complex of a graph G can be used to establish a lower bound of its chromatic number $\chi(G)$ and that this bound can be used to prove Kneser's conjecture. The following diagram indicates how Lovász' result for Kneser graphs has been generalised in the subsequent years to Ziegler's result for s-disjoint r-uniform Kneser hypergraphs $\mathrm{KG}_s^r\mathcal{T}$ associated to a family $\mathcal{T} \subseteq 2^{[n]}$ of subsets of an n-set $[n]$.

$$
\begin{tikzcd}
\begin{array}{c}\text{Lovász [7]}\\ \mathrm{cd}_1^2\binom{[n]}{k} \leq \chi(\mathrm{KG}_1^2\binom{[n]}{k})\end{array} \arrow[r] \arrow[d] & \begin{array}{c}\text{Alon, Frankl, Lovász [1]}\\ \mathrm{cd}_1^r\binom{[n]}{k} \leq (r-1)\cdot\chi(\mathrm{KG}_1^r\binom{[n]}{k})\end{array} \arrow[r] \arrow[d] & \begin{array}{c}\text{Sarkaria [12]}\\ \mathrm{cd}_s^r\binom{[n]}{k} \leq (r-1)\cdot\chi(\mathrm{KG}_s^r\binom{[n]}{k})\end{array} \arrow[d] \\
\begin{array}{c}\text{Dol'nikov [3]}\\ \mathrm{cd}_1^2\mathcal{T} \leq \chi(\mathrm{KG}_1^2\mathcal{T})\end{array} \arrow[r] & \begin{array}{c}\text{Kříž [5, 6]}\\ \mathrm{cd}_1^r\mathcal{T} \leq (r-1)\cdot\chi(\mathrm{KG}_1^r\mathcal{T})\end{array} \arrow[r] & \begin{array}{c}\text{Ziegler [14]}\\ \mathrm{cd}_s^r\mathcal{T} \leq (r-1)\cdot\chi(\mathrm{KG}_s^r\mathcal{T})\end{array}
\end{tikzcd}
$$

We defer precise definitions of the s-disjoint r-colourability defect $\mathrm{cd}_s^r\mathcal{T}$ and of $\mathrm{KG}_s^r\mathcal{T}$ to Section 2. For $s > 1$, the Kneser hypergraph $\mathrm{KG}_s^r\mathcal{T}$ just mentioned is not a hypergraph in the sense of Berge [2]. It is a hypergraph *with multiplicities*, that is, a hyperedge may contain multiple copies of nodes but must contain at least two different nodes. Usually a hyperedge of a hypergraph consists of distinct nodes, i.e. the hypergraph is *multiplicity-free*. In Section 2 we also define multiplicity-free Kneser hypergraphs $\mathrm{kg}_s^r\mathcal{T}$ (a hypergraph in the sense of Berge) and point out that the proof of Theorem 5.1 of Ziegler [14] does not work in the generality stated there. Section 3 is devoted to examples. We show that neither Sarkaria's nor Ziegler's result holds if we replace $\mathrm{KG}_s^r\mathcal{T}$ by $\mathrm{kg}_s^r\mathcal{T}$.

In Section 5 we extend a result from Alon, Frankl, and Lovász [1] for multiplicity-free r-uniform hypergraphs : We give a topolgical lower bound of the chromatic number of an r-uniform hypergraph with or without multiplicities if r is the power of a prime. A review of known facts about free and fixed-point free group actions and simplicial complexes is given in Section 4. We also describe a variety of simplicial complexes that we need later, such as a box complex associated to an arbitrary r-uniform hypergraph $\mathcal{S}$. This complex is different from (but closely related to) the complex considered by Alon, Frankl, and Lovász. In an unpublished preprint from 1987, Özaydin extended their result to the case where r is the power of a prime, [11].

In Section 6 we give a new proof of Ziegler's result for r-uniform s-disjoint Kneser hypergraphs with multiplicities. The main tool used is "Sarkaria's inequality". The proof is inspired by Matoušek's proof [8] of the result of Kříž [5, 6]. In fact we prove a bit more: The colourability defect of a set system $\mathcal{T}$ is not the only lower bound for the chromatic number of the r-uniform

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Kneser hypergraph associated to $\mathcal{T}$. If r is prime we squeeze between these two numbers the index of an associated simplicial complex. Theoretically, this might yield better lower bounds but has not been investigated so far.

For arbitrary r-uniform hypergraphs with or without multiplicities, no lower bound except the topological one stated in Theorem 2 is known. The problem one is confronted with is the same that one faces in proving the topological Tverberg theorem. Only partial results in case of prime-powers are achieved so far. Moreover, it is not known whether every r-uniform hypergraph (multiplicity-free or not) can be realised as an r-uniform s-disjoint Kneser hypergraph for some s and an appropriate set system. A result of this type is known for graphs: Every graph is a Kneser graph, [10].

2. Preliminaries

s-disjoint sets. For integers $r > s \geq 1$ we say that the subsets $S_1, \ldots, S_r$ of $[n]$ are *s-disjoint* if each element of $[n]$ occurs in at most s of the sets S_i, or equivalently, if the intersection of any choice of $s+1$ sets is empty. This formulation is the reason that this concept is called s-wise disjoint by Sarkaria [12]. We emphazise that $S_i = S_j$ may occur for $i \neq j$.

s-disjoint r-colourability defect. The *s-disjoint r-colourability defect* $\mathrm{cd}_s^r \mathcal{S}$ of $\mathcal{S} \subseteq 2^{[n]} \setminus \emptyset$ is the number of elements one has to remove from the multiset $[n]^s$ such that the remaining multiset can be covered by an s-disjoint r-tuple of sets such that none of the sets contains an element from $\mathcal{S}$. This number can be computed by evaluating

$$\mathrm{cd}_s^r \mathcal{S} = n \cdot s - \max \left\{ \sum_{j=1}^{r} |R_j| \;\middle|\; \begin{array}{c} R_1, \ldots, R_r \subseteq [n] \text{ } s\text{-disjoint} \\ \text{and } S \notin R_j \text{ for all } S \in \mathcal{S} \text{ and all } j \end{array} \right\}.$$

r-multisubsets of [n]. The collection $x_1, \ldots, x_r$ of elements of $[n]$ is called an *r-multisubset* of $[n]$ if the set $\{x_1, \ldots, x_r\}$ has cardinality at least two. We denote an r-multiset by $\{\{x_1, \ldots, x_r\}\}$.

r-uniform hypergraphs with or without multiplicities. Consider $\mathcal{S} \subseteq 2^{[n]}$ such that $\bigcup_{S \in \mathcal{S}} S = [n]$. An r-uniform hypergraph $\mathcal{S}$ without multiplicities coincides with Berge's definition of an r-uniform hypergraph, [2]: The vertices are $[n]$ and the hyperedges are the r-subsets $\mathcal{S}$ of $[n]$. Let $\mathcal{S}'$ be a set of r-multisubsets of $[n]$ such that for every $i \in [n]$ exists an $S \in \mathcal{S}'$ with $i \in S$. We call $\mathcal{S}'$ an *r-uniform hypergraph with multiplicities* and say that it has *node set* $\mathrm{V}(\mathcal{S}') = [n]$ and *hyperedge set* $\mathrm{E}(\mathcal{S}') = \mathcal{S}'$.

r-uniform s-disjoint Kneser hypergraphs. For a set $\mathcal{T} = \{T_1, \ldots, T_m\}$ of subsets of $[n]$, we consider the *r-uniform s-disjoint Kneser hypergraph* $\mathrm{KG}_s^r \mathcal{T}$ *with multiplicities* on the node set $\mathrm{V}(\mathrm{KG}_s^r \mathcal{T}) = [m] = [|\mathcal{T}|]$ with hyperedges

$$\mathrm{E}(\mathrm{KG}_s^r \mathcal{T}) := \left\{ \{\{k_1, \ldots, k_r\}\} \;\middle|\; \begin{array}{c} k_i \in [m],\ \{\{k_1, \ldots, k_r\}\} \text{ is an } r\text{-multiset,} \\ \text{and } T_{k_1}, \ldots, T_{k_r} \text{ are } s\text{-disjoint} \end{array} \right\}.$$

The *r-uniform s-disjoint Kneser hypergraph* $\mathrm{kg}_s^r \mathcal{T}$ *without multiplicities* has the same node set $\mathrm{V}(\mathrm{kg}_s^r \mathcal{T}) = [m] = [|\mathcal{T}|]$ but a different hyperedge set:

$$\mathrm{E}(\mathrm{kg}_s^r \mathcal{T}) := \left\{ \{k_1, \ldots, k_r\} \subseteq [m] \;\middle|\; \begin{array}{c} \{k_1, \ldots, k_r\} \text{ is an } r\text{-set} \\ \text{and } T_{k_1}, \ldots, T_{k_r} \text{ are } s\text{-disjoint} \end{array} \right\}.$$

In the special case $s = 1$ we have $\mathrm{KG}_s^r \mathcal{T} = \mathrm{kg}_s^r \mathcal{T}$ since an r-multiset with r pairwise disjoint elements can be seen as an r-set. In particular for $r = 2$ we have $s = 1$ and both definitions specialise to a Kneser graph of $\mathcal{T} \subseteq 2^{[n]}$.

Colourings. A colouring of an r-uniform hypergraph $\mathcal{S}$ with m colours is a map $c : \mathrm{V}(\mathcal{S}) \to [m]$ that assigns to each node of $\mathcal{S}$ a colour so that no hyperedge is monochromatic, that is, for $e \in \mathrm{E}(\mathcal{S})$ we have $|\{c(x) \mid x \in e\}| \geq 2$. The *chromatic number* $\chi(\mathcal{S})$ is the smallest number m such that a colouring of $\mathcal{S}$ with m colours exists. Every hyperedge of $\mathrm{kg}_s^r \mathcal{T}$ is a hyperedge of $\mathrm{KG}_s^r \mathcal{S}$, hence:

$$\chi(\mathrm{kg}_s^r \mathcal{T}) \leq \chi(\mathrm{KG}_s^r \mathcal{T}).$$

A remark on Theorem 5.1 of [14]. Ziegler states his Theorem 5.1 for hypergraphs in the sense of Berge. The proof on page 679 of [14] yields the disired contradiction only if one assumes a

colouring of a hypergraph with r-multisets as hyperedges. More precisely, the construction only guarantees that the p subsets $S_1, \ldots, S_p$ of $[n]$ are s-disjoint, they need not to be pairwise different.

3. Examples and Counterexamples

Example 1. We give an example to illustrate the two concepts of s-disjoint r-uniform Kneser hypergraphs and to see that the chromatic numbers $\chi(\mathrm{kg}_s^r\mathcal{S})$ and $\chi(\mathrm{KG}_s^r\mathcal{S})$ can be different. We restrict ourselves to the following small but interesting case: $r = 3$, $s = 2$, and $\mathcal{T} \subseteq \binom{[5]}{2}$. Let us consider

$$\mathcal{T} := \{\ \{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{2,3\}, \{4,5\}\ \}.$$

The 3-uniform 2-disjoint Kneser hypergraph $\mathrm{kg}_2^3\mathcal{T}$ without multiplicities has 6 nodes and any hyperedge consists either of two nodes of type $\{1,x\}$ plus $\{2,3\}$ or $\{4,5\}$ or it consists of $\{2,3\}$, $\{4,5\}$ plus one node of type $\{1,x\}$. Colouring $\mathrm{kg}_s^r\mathcal{T}$ means therefore colouring the edges of the graph shown in Figure 1 such that no triple of edges that form a hyperedge is monochromatic. This can be done with 2 colours as indicated in Figure 1: Colour each node that contains 1 with colour one, and colour $\{2,3\}$ and $\{4,5\}$ with the other colour. Among others, the s-disjoint r-uniform Kneser hypergraph $\mathrm{KG}_s^r\mathcal{T}$ with multiplicities has the following additional hyperedges:

$$\{\{\ \{2,3\}, \{2,3\}, \{4,5\}\ \}\} \qquad \text{and} \qquad \{\{\ \{2,3\}, \{4,5\}, \{4,5\}\ \}\}.$$

These hyperedges force us to use at least three colours; for a colouring see also Figure 1. The sets $R_1 = \{2,4\}$, $R_2 = \{2,5\}$ and $R_3 = \{3,4\}$ are 2-disjoint, i.e. $\mathrm{cd}_s^r\mathcal{T} \leq 4$. From Ziegler's theorem or our Theorem 4, we know that $\frac{\mathrm{cd}_2^3\mathcal{T}}{3-1} \leq \chi(\mathrm{KG}_2^3\mathcal{T})$. In this particular example we have

$$\frac{\mathrm{cd}_2^3\mathcal{T}}{3-1} \leq 2 = \chi(\mathrm{kg}_2^3\mathcal{T}) < \chi(\mathrm{KG}_2^3\mathcal{T}) = 3.$$

Counterexample 1. Example 1 can be modified easily to show that the colourability defect is not a lower bound for $\mathrm{kg}_{r-2}^r\mathcal{T}$ in general. For fixed $n \geq 5$, we consider the following set $\mathcal{T}$ of subsets of $[n]$:

$$\mathcal{T} := \{\ \{1,2\}, \ldots, \{1,n\}, \{2,3\}, \{4,5\}\ \}\,.$$

The $(r-2)$-disjoint r-uniform Kneser hypergraph $\mathrm{kg}_{r-2}^r\mathcal{T}$ is easily described. Every hyperedge contains $(r-2)$ different elements of $\{\ \{1,2\}, \ldots, \{1,n\}\ \}$ plus $\{2,3\}$ and $\{4,5\}$, so $\chi(\mathrm{kg}_{r-2}^r\mathcal{T}) = 2$ if $n \geq r-1$. To compute the $(r-2)$-disjoint r-colourability defect of $\mathcal{T}$, we cover $[n]^{r-2}$ by r sets so that no set contains an element of $\mathcal{T}$. Obviously, a set that contains 1 does not contain any other element. Let r_1 denote the number of sets that contain 1. The sets that do not contain 1 cannot contain 2 and 3 as well as 4 and 5 at the same time. There are $r_2 = r - r_1$ such sets. Therefore, we have not covered $(r-2) - r_1$ copies of 1, $2(r-2) - r_2$ copies of 2 or 3, and $2(r-2) - r_2$ copies of 4 or 5. In other words, at least $(r-2) - r_1 + 2(r-2) - r_2 + 2(r-2) - r_2 = 3n - 10$ elements are not covered or $\mathrm{cd}_s^r\mathcal{T} \geq 3r - 10$. For $r > 8$ this implies $\mathrm{cd}_{r-2}^r\mathcal{T} > (r-1)\chi(\mathrm{kg}_{r-2}^r\mathcal{T})$.

Counterexample 2. We now show that the colourability defect is not even a lower bound for $\mathrm{kg}_{r-1}^r\binom{[n]}{2}$ in general. From Ziegler [14, Lemma 3.2], we know $\mathrm{cd}_s^r\binom{[n]}{2} = \max\{ns - r(k-1), 0\}$,

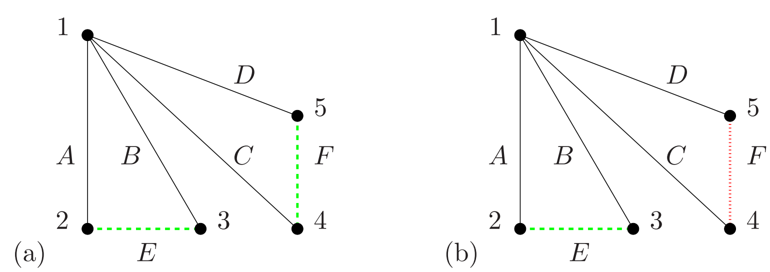

Figure 1. The edges A, B, C, D, E, and F represent the nodes of $\mathrm{kg}_s^r\mathcal{T}$ and $\mathrm{KG}_s^r\mathcal{T}$. In (a) the nodes of the hypergraph $\mathrm{kg}_s^r\mathcal{T}$ are coloured with two colours (best possible). In (b) The nodes of the hypergraph $\mathrm{KG}_s^r\mathcal{T}$ are coloured with three colours (best possible).

hence we have to show that $(r-1)\chi(\mathrm{kg}^r_{r-1}\binom{[n]}{2}) < (r-1)n-r$. It suffices to colour $\mathrm{kg}^r_s\binom{[n]}{2}$ with $n-2$ colours. This can be done by a greedy colouring already known to Kneser in case of graphs: Assign colour i to $T \in \mathcal{T}$ if i is the smallest element of T and $i \le n-3$. The elements not coloured yet are $\{n-2, n-1\}$, $\{n-2, n\}$, and $\{n-1, n\}$; too few to form a hyperedge. We colour these elements by colour $n-2$. This is in general not optimal, but suffices.

4. Groups acting on simplicial complexes

This section summarises standard definitions and known facts. For many examples and a detailed treatment we refer to Matoušek's textbook, [9].

Simplicial complexes. An *abstract simplicial complex* K is a finite hereditary set system with vertex set $\mathrm{V}(\mathsf{K})$. For sets $A_1, \ldots, A_t$ we define $A_1 \uplus \ldots \uplus A_t := \{(a,1) \mid a \in A_1\} \cup \ldots \cup \{(a,t) \mid a \in A_t\}$. An important construction in the category of simplicial complexes is the *deleted join operation*. For a simplicial complex K and positive integers $r \ge s$ the *r-fold s-wise deleted join* $\mathsf{K}^{*r}_{\Delta(s)}$ is defined as

$$\mathsf{K}^{*r}_{\Delta(s)} := \{\mathsf{F}_1 \uplus \ldots \uplus \mathsf{F}_r \mid \mathsf{F}_i \in \mathsf{K} \text{ and } \mathsf{F}_1, \ldots, \mathsf{F}_r \text{ is } s\text{-wise disjoint }\}.$$

To avoid confusion, we point to the fact that s indicates s-wise disjointness in the definition not s-disjointness. Any abstract simplicial complex K can be realized as a topological space $\|\mathsf{K}\|$ in $\mathbb{R}^d$ for some d.

Free $\mathbb{Z}_r$-spaces and $\mathbb{Z}_r$-index. A *free $\mathbb{Z}_r$-space* is a topological space X together with a free $\mathbb{Z}_r$-action Φ, i.e. for all $g, h \in \mathbb{Z}_r$ we have $\Phi(g) \circ \Phi(h) = \Phi(g+h)$, $\Phi(0) = \mathrm{Id}$, and $\Phi(g)$ has no fixed point for $g \in \mathbb{Z}_r \setminus \{0\}$. A continuous map f between $\mathbb{Z}_r$-spaces (X, Φ_X) and (Y, Φ_Y) is $\mathbb{Z}_r$-*equivariant* (or a *$\mathbb{Z}_r$-map* for simplicity) if f commutes with the $\mathbb{Z}_r$-actions, i.e. $f \circ \Phi_X = \Phi_Y \circ f$. A simplicial complex (K, Φ) is a *free simplicial $\mathbb{Z}_r$-space* (or a *free simplicial $\mathbb{Z}_r$-complex*) if $\Phi : \mathsf{K} \to \mathsf{K}$ is a simplicial map such that $\|\Phi\|$ is a free $\mathbb{Z}_r$-action on $\|\mathsf{K}\|$. A *simplicial $\mathbb{Z}_r$-equivariant* map f is a simplicial map between two simplicial $\mathbb{Z}_r$-spaces that commutes with the free $\mathbb{Z}_r$-actions. An important class of free $\mathbb{Z}_r$-spaces is called $E_n\mathbb{Z}_r$-space: A free $\mathbb{Z}_r$-space is an $E_n\mathbb{Z}_r$-space if it is n-dimensional and $(n-1)$-connected. The $\mathbb{Z}_r$-index $\mathrm{ind}_{\mathbb{Z}_r}(X)$ of (X, Φ) is the smallest n such that a $\mathbb{Z}_r$-map from X to some $E_n\mathbb{Z}_r$-space exists. A generalised Borsuk–Ulam theorem, e.g. Dold's theorem [4] for free group actions provides the index for $E_n\mathbb{Z}_r$-spaces: There is no $\mathbb{Z}_r$-map from $E_n\mathbb{Z}_r$ to $E_{n-1}\mathbb{Z}_r$. Since we consider cyclic group actions most of the time, we tend to refer to a $\mathbb{Z}_r$-space X without explicit reference to Φ.

Sarkaria's inequality. A useful inequality concerning the $\mathbb{Z}_r$-index of the join $\mathsf{K} * \mathsf{L}$ of two free simplicial $\mathbb{Z}_r$-complexes K and L is *Sarkaria's inequality*, [8, 9]:

$$\mathrm{ind}_{\mathbb{Z}_r}(\mathsf{K} * \mathsf{L}) \le \mathrm{ind}_{\mathbb{Z}_r}(\mathsf{K}) + \mathrm{ind}_{\mathbb{Z}_r}(\mathsf{L}) + 1.$$

Examples: $\mathsf{P}^{\mathbf{r}}_{\mathbf{s}}$ and $\mathsf{P}^{\mathbf{r}}_{\mathbf{s}}\mathcal{S}$. Let $1 \le s < r$ where r is a prime. Consider the poset P^r_s of s-disjoint r-tuples $(S_1, \ldots, S_r)$ of subsets of $[n]$ with $\bigcup_{i \in [r]} S_i \ne \emptyset$ ordered by componentwise inclusion, that is, $(S_1, \ldots, S_r) \le (T_1, \ldots, T_r)$ if $S_i \subseteq T_i$ for all $1 \le i \le r$. The order complex of P^r_s is denoted by P^r_s and can be interpreted as the barycentric subdivision of the r-fold $(s+1)$-wise deleted join of an $(n-1)$-simplex σ^{n-1}, which is homotopy equivalent to a wedge of $(ns-1)$-dimensional spheres, [9] or alternatively [12]. Hence P^r_s is an $E_{ns-1}\mathbb{Z}_r$-space. Another space we shall need later is the order complex $\mathsf{P}^r_s\mathcal{S}$ of the subposet $P^r_s\mathcal{S}$ of P^r_s that consists only of those r-tuples $(S_1, \ldots, S_r)$ with $\sum_{i=1}^r |S_i| \ge ns - \mathrm{cd}^r_s\mathcal{S} + 1$, where $\mathcal{S} \subseteq 2^{[n]}$. Such tuples have the useful property that there exists an $S \in \mathcal{S}$ such that $S \subseteq S_i$ for some i.

Fixed-point free actions. A *fixed-point free action* Φ of a group G of order r on a topological space X means that no $x \in X$ is fixed by all $g \in G$. Obviously, a free group action is also fixed-point free. A standard example is $(\mathbb{R}^m)^r$ with the diagonal $\{(v, \ldots, v) \mid v \in \mathbb{R}^m\}$ removed. We have a standard action of G on $(\mathbb{R}^m)^r$ by permuting the r copies of $\mathbb{R}^m$. The action is fixed-point free on $(\mathbb{R}^m)^r \setminus \{(v, \ldots, v) \mid v \in \mathbb{R}^m\}$ for all r. It is free if r is a prime. The space $(\mathbb{R}^m)^r \setminus \{(v, \ldots, v) \mid v \in \mathbb{R}^m\}$ is homotopy equivalent to an $(m(r-1)-1)$-sphere.

Theorem 1 ("Volovikov's theorem", [13])**.**
Let $r = p^t$ be a power of a prime and consider a fixed-point free action of $(\mathbb{Z}_p)^t$ on X and Y.

Suppose that for all $i \leq \ell$ we have $\widetilde{H}^i(X;\mathbb{Z}_p) = 0$ and that Y is a finite-dimensional cohomology k-dimensional sphere over the field $\mathbb{Z}_p$.
If there exists a $(\mathbb{Z}_r)^t$-equivariant map $f : X \longrightarrow Y$, then $\ell < k$.

Box complexes. Alon, Frankl, and Lovász [1], Kříž [5], and Matoušek and Ziegler [10] describe different versions of a box complex to obtain topological lower bounds for a (hyper)graph. We now define a box complex $\mathsf{B}_0(\mathcal{S})$ associated to an r-uniform hypergraph $\mathcal{S}$ which reduces in the graph case to the box complexes $\mathsf{B}_0(G)$ described by Matoušek and Ziegler [10]. This complex is different from the complexes considered by Alon, Frankl, and Lovász [1] and Kříž [5]. For an r-uniform hypergraph $\mathcal{S}$ without multiplicities we define

$$\mathsf{B}_0(\mathcal{S}) := \left\{ (U_1, \ldots, U_r) \subseteq \mathrm{V}(\mathcal{S})^r \;\middle|\; \begin{array}{c} \bigcup_{i\in[r]} U_i \neq \emptyset \text{ and if all } U_i \neq \emptyset \text{ we have:} \\ u_i \in U_i \ (1 \leq i \leq r) \text{ implies } \{u_1, \ldots, u_r\} \in \mathrm{E}(\mathcal{S}) \end{array} \right\}.$$

In case of an r-uniform hypergraph with multiplicities we replace the r-sets in the definition by r-multisubsets. The box complex of an r-uniform hypergraph has a free action by cyclic shift if r is prime and a fixed-point free action for arbitrary r. Since $\mathrm{kg}^r_s\mathcal{S}$ (considered as a hypergraph with multiplicities) is a subhypergraph of $\mathrm{KG}^r_s\mathcal{S}$ we have for prime r:

$$\mathrm{ind}_{\mathbb{Z}_r}(\mathsf{B}_0(\mathrm{kg}^r_s\mathcal{S})) \leq \mathrm{ind}_{\mathbb{Z}_r}(\mathsf{B}_0(\mathrm{KG}^r_s\mathcal{S})).$$

Colour complexes. The colour complex $\mathcal{C}$ of an r-uniform hypergraph $\mathcal{S}$ is the simplicial complex $(\sigma^{\chi(\mathcal{S})-1})^{*r}_{\Delta(r)}$, i.e. the vertex set $\mathrm{V}(\mathcal{C})$ consists of r copies of $[\chi(\mathcal{S})]$ and the simplices are ordered r-tuples $(C_1, \ldots, C_r)$ of subsets of $[\chi(\mathcal{S})]$ with $\bigcup_{i\in[n]} C_i \neq \emptyset$ and $\bigcap_{j=1}^r C_j = \emptyset$. The colour complex is free if r is prime.

5. A topological lower bound for the chromatic number

Theorem 2. *Let $r = p^t$ for a prime p and a positive integer t. Consider an r-uniform hypergraph $\mathcal{S}$ with or without multiplicities. Suppose that $\widetilde{H}^i(\mathsf{B}_0(\mathcal{S});\mathbb{Z}_p) = 0$ for $i \leq \ell$. Then*

$$\chi(\mathcal{S}) \geq \left\lceil \frac{\ell+2}{r-1} \right\rceil.$$

Proof. A colouring $c : \mathrm{V}(\mathcal{S}) \longrightarrow [\chi(\mathcal{S})]$ induces a continuous $\mathbb{Z}_r$-map

$$f_c : \mathsf{B}_0(\mathcal{S}) \longrightarrow (\mathbb{R}^{\chi(\mathcal{S})})^r \setminus \{(v, \ldots, v) \mid v \in \mathbb{R}^{\chi(\mathcal{S})}\}.$$

Consider the standard basis $e_1, \ldots, e_{r\cdot\chi(\mathcal{S})}$. Map a vertex $(\emptyset, \ldots, \emptyset, v, \emptyset, \ldots, \emptyset)$ that has non-empty coordinate j to $e_{j\cdot c(v)}$, and extend by linearity. The image of this map is contained in the boundary of the simplex that is spanned by $e_1, \ldots, e_{r\cdot\chi(\mathcal{S})}$ and does not meet $\{(v, \ldots, v) \mid v \in \mathbb{R}^{\chi(\mathcal{S})}\}$. In particular, a $(\mathbb{Z}_p)^t$-homotopic copy of $\mathrm{Im}\, f_c$ is contained in a sphere of dimension $((r-1)\cdot\chi(\mathcal{S})-1)$ by normalising each point of $\mathrm{Im}\, f_c$. The spaces $\mathsf{B}_0(\mathcal{S})$ and $\mathrm{Im}\, f_c$ (as well its homotopic copy) are fixed-point free $(\mathbb{Z}_p)^t$-spaces, hence we can apply Volovikov's theorem (Theorem 1) to deduce $\ell < (r-1)\cdot\chi(\mathcal{S}) - 1$. ☐

So far it is not possible to prove this result for arbitrary r. In Section 6, we only need the following weaker statement. We include its proof since we can avoid Volovikov's theorem.

Theorem 3. *Let r be a prime and $\mathcal{S}$ be an r-uniform hypergraph with or without multiplicities. Then*

$$\chi(\mathcal{S}) \geq \left\lceil \frac{\mathrm{ind}_{\mathbb{Z}_r}(\mathsf{B}_0(\mathcal{S})) + 1}{r-1} \right\rceil.$$

Proof. We have to show that $\mathrm{ind}_{\mathbb{Z}_r}(\mathsf{B}_0(\mathcal{S})) \leq (r-1)\cdot\chi(\mathcal{S}) - 1$. A colouring $c : \mathrm{V}(\mathcal{S}) \longrightarrow [\chi(\mathcal{S})]$ induces a map $f_c : \mathsf{B}_0(\mathcal{S}) \longrightarrow \mathcal{C}$ defined on the vertices by

$$(\emptyset, \ldots, \emptyset, v, \emptyset, \ldots, \emptyset) \longmapsto (\emptyset, \ldots, \emptyset, c(v), \emptyset, \ldots, \emptyset),$$

where $v \in \mathrm{V}(\mathcal{S})$ and a vertex of $\mathsf{B}_0(\mathcal{S})$ that has non-empty entry in coordinate i is mapped to a vertex of $\mathcal{C}$ that has non-empty entry in coordinate i. This map is well-defined since c is a

colouring, it extends naturally to a simplicial map, and it is $\mathbb{Z}_r$-equivariant, no matter whether $\mathcal{S}$ is multiplicity-free or not. Hence $\mathrm{Im}\, f_c$ is a simplicial $\mathbb{Z}_r$ subcomplex of $\mathcal{C}$ and we have

$$\mathrm{ind}_{\mathbb{Z}_r}(\mathsf{B}_0(\mathcal{S})) \leq \mathrm{ind}_{\mathbb{Z}_r}(\mathrm{Im}\, f_c) \leq \dim(\mathrm{Im}\, f_c) \leq (r-1)\cdot\chi(\mathcal{S}) - 1$$

since a maximal simplex of $\mathrm{Im}\, f_c$ contains at most $(r-1)\cdot\chi(\mathcal{S})$ many vertices. □

6. A combinatorial lower bound for the chromatic number of Kneser hypergraphs with multiplicities

Theorem 4. *For integers $1 \leq s < r$ and a family $\mathcal{T}$ of subsets of $[n]$, we have*

$$\chi(\mathrm{KG}^r_s\mathcal{T}) \geq \left\lceil \frac{\mathrm{cd}^r_s\mathcal{T}}{r-1} \right\rceil .$$

If r is prime we have

$$\chi(\mathrm{KG}^r_s\mathcal{T}) \geq \left\lceil \frac{\mathrm{ind}_{\mathbb{Z}_r}(\mathsf{B}_0(\mathrm{KG}^r_s\mathcal{T}))}{r-1} \right\rceil \geq \left\lceil \frac{\mathrm{cd}^r_s\mathcal{T}}{r-1} \right\rceil .$$

The proof consists of two steps: First one has to prove $\mathrm{cd}^r_s\mathcal{T} \leq \mathrm{ind}_{\mathbb{Z}_r}(\mathsf{B}_0(\mathrm{KG}^r_s\mathcal{T}))$ for all prime numbers r, then the claim follows by induction from Theorem 3 as shown by Sarkaria [12] or alternatively Ziegler [14]. The idea of this induction traces back to Alon, Frankl, and Lovász [1] and was also applied by Kříž [6] and Matoušek [8].

Proof. For a subset $U \subseteq [n]$ we define a map $g : 2^{[n]} \longrightarrow 2^{\mathcal{T}}$ by $U \longmapsto \{T \in \mathcal{T} \mid T \subseteq U\}$. This map is used to define another map $f : \mathsf{P}^r_s\mathcal{T} \longrightarrow \mathrm{sd}(\mathsf{B}_0(\mathrm{KG}^r_s\mathcal{T}))$ via $(U_1,\ldots,U_r) \longmapsto (g(U_1),\ldots,g(U_r))$. This map is well-defined since at least one U_i contains an element of $\mathcal{T}$ and $U'_1,\ldots,U'_r$ are s-disjoint if $U'_i \subseteq U_i$ and $U_1,\ldots,U_r$ is s-disjoint. The map yields a simplicial map because chains of elements of P^r_s are mapped to chains of simplices of $\mathsf{B}_0(\mathrm{KG}^r_S\mathcal{T})$. Finally, the map is $\mathbb{Z}_r$-equivariant and surjective, hence

$$\mathrm{ind}_{\mathbb{Z}_r}(\mathsf{P}^r_s\mathcal{T}) \leq \mathrm{ind}_{\mathbb{Z}_r}(\mathrm{Im}\, f) = \mathrm{ind}_{\mathbb{Z}_r}(\mathrm{sd}(\mathsf{B}_0(\mathrm{KG}^r_s\mathcal{T}))) = \mathrm{ind}_{\mathbb{Z}_r}(\mathsf{B}_0(\mathrm{KG}^r_s\mathcal{T})).$$

To apply Sarkaria's inequality, consider the subcomplex L of P^r_s that is induced from the vertices $\mathrm{V}(\mathsf{P}^r_s) \setminus \mathrm{V}(\mathsf{P}^r_s\mathcal{T})$ and use that $\mathsf{P}^r_s \subseteq \mathsf{P}^r_s\mathcal{T} * \mathsf{L}$. Hence

$$\mathrm{ind}_{\mathbb{Z}_r}(\mathsf{P}^r_s) \leq \mathrm{ind}_{\mathbb{Z}_r}(\mathsf{P}^r_s\mathcal{T} * \mathsf{L}) \leq \mathrm{ind}_{\mathbb{Z}_r}(\mathsf{P}^r_s\mathcal{T}) + \mathrm{ind}_{\mathbb{Z}_r}(\mathsf{L}) + 1.$$

Since $\mathrm{ind}_{\mathbb{Z}_r}(\mathsf{P}^r_s) = ns - 1$ and since the dimension is an upper bound for the index, we have

$$ns - 1 - \dim(\mathsf{L}) - 1 \leq \mathrm{ind}_{\mathbb{Z}_r}(\mathsf{P}^r_s\mathcal{T}).$$

But $\dim(\mathsf{L}) \leq ns - \mathrm{cd}^r_s\mathcal{T} - 1$ since every chain of length larger than $ns - \mathrm{cd}^r_s\mathcal{T}$ in P^r_s contains at least one s-disjoint r-tuple $(U_1,\ldots,U_r)$ that satisfies $\sum_{i=1} r|U_i| \geq ns - \mathrm{cd}^r_s\mathcal{T} + 1$. □

We could hide the box complex in the proof and compose $f : \mathsf{P}^r_s\mathcal{T} \longrightarrow \mathrm{sd}(\mathsf{B}_0(\mathrm{KG}^r_s\mathcal{T}))$ with the "barycentric subdivision" $\mathrm{sd}(f_c) : \mathrm{sd}(\mathsf{B}_0(\mathcal{S})) \longrightarrow \mathrm{sd}(\mathcal{C})$ to obtain $(r-1)\cdot\chi(\mathrm{KG}^r_s\mathcal{T}) \geq \mathrm{cd}^r_s\mathcal{T}$.

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